SUMMARY REPORT

IMPROVEMENTS TO NASTRAN

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ABSTRACT

Under NASA Contract NAS 1-10917, Bell Aerospace was responsible for improvements to the operational capabilities of NASTRAN. The improvements included the development and implementation of a number of new finite elements into the existing NASTRAN element library and also the development of a finite element heat transfer analysis capability.

The present report summarizes these additions to NASTRAN.

TABLE OF CONTENTS

I. -   INTRODUCTION

Under NASA Contract NAS 1-10917, Bell Aerospace Company was responsible for improving the operational capabilities of the NASA general purpose finite element analysis program NASTRAN. The improvements under this activity were to be conducted in two general and related areas. The first area comprised the development and implementation of a number of new finite elements into the existing NASTRAN element library.

The second major task was the development of a finite element heat transfer analysis capability within the framework of the present NASTRAN program structure. This thermal analysis capability includes conduction, convection and radiation and is capable of handling both steady-state and transient analysis conditions. The development of the finite element heat transfer analysis capability was subcontracted by Bell Aerospace to the MacNeal-Schwendler Corporation, while the finite element development was performed by Bell as the prime contractor.

The new finite elements and the heat transfer analysis capabilities were developed, coded and checked out using demonstration problems supplied by the NASTRAN System Management Office (NSMO). Upon completion the elements were delivered along with full documentation to NSMO for implementation into NASTRAN. The documentation accompanying each element and the heat transfer capability consisted of a full technical derivation which were prepared as new sections of the NASTRAN Theoretical Manual. In addition, complete updates to the NASTRAN User's and Programmer's Manuals were supplied documenting the many changes introduced through the implementation of the improved analysis capabilities.

The full derivations of all elements are presented in the new sections of the Theoretical Manual. The purposes of the present report is to summarize the new finite elements and heat transfer analysis capabilities in NASTRAN.

In addition to the six new structural elements coded into NASTRAN, a number of other elements were developed to various levels but were not coded. For these elements preliminary drafts of the Theoretical Manual Entries were prepared but these will not appear in the published Theoretical Manual. Summaries of theoretical derivations of these elements are also included in the present report.

II. -   SUMMARY OF IMPROVEMENTS TO NASTRAN

   A.   NEW ELEMENTS IN NASTRAN

   (i)   General Element

        The general element is a structural stiffness el-
ement connected to any number of degrees of freedom,
as specified by the user. In defining the form of the
externally generated data on the stiffness of the ele-
ment, two major options are provided.

        (a) The user supplies the deflection influence
coefficients (flexibility matrix) for the structure
supported in a non-redundant manner. The associated
matrix of the restrained rigid body modes may be in-
put or may be generated internally by the program.
One use of the general element is the representation
of part of a structure by means of externally measured
data. Such data are more usually generated in the form
of flexibilities than direct stiffnesses.

        (b) The stiffness matrix of the element may
be input directly. This stiffness matrix may be for
an unsupported body, containing all the rigid body modes,
or boundary conditions may already have been imposed
resulting in the deletion of some or all of the rigid
body modes. In the latter case, the option is given
for automatic inflation of the stiffness matrix to
reintroduce the restrained rigid body terms, provided
that the original support conditions did not constitute
a redundant set of reactions.

        The former capability of handling an input flex-
ibility matrix was already resident in the NASTRAN
system. The ability to input and process a stiffness

matrix was developed as part of the current contract.

While the direct input and use of an externally gen-
erated stiffness matrix is straightforward, the ability
to regenerate some or all of the suppressed rigid body
modes requires special considerations.

The matrix relating the constrained degrees of freedom
of the general element to a reference set of motions
at the origin of the basic coordinate system may be
rectangular and non-invertible. To overcome this dif-
ficulty the matrix is partitioned using a rank tech-
nique and inflated by addition of an identity matrix.
The appropriate columns of the inverted inflated ma-
trix are then used to generate the necessary rigid body
modes for the supported general element.

The other stages in the theoretical basis of the new
element follow the existing pattern for the flexibility
based general element.

(ii) Rigid Body Element
The rigid element is a generalization of the multi-
point constraint capability resident in NASTRAN. The
element procedures automatically generate the coeffi-
cients which define the linear dependency of the linked
degrees of freedom instead of their being supplied di-
rectly by the user.

The rigid element may be a one-, or three-dimensional
structural element with any number of grid points. At
these points, any or all degrees of freedom may be con-
nected to the remaining structure as specified by the
user. The only motions that the element is permitted

4

to have are six gross rigid body degrees of freedom.
The element has the effect of providing automatic
coupling of a large number of component degrees of
freedom at various grid points and condensing these to
the six degrees of freedom associated with the motion
of the rigid body. In the absence of boundary con-
straints the motion of the element will be fully de-
fined by the six degrees of freedom at a reference
point. This reference point may be either a struct-
ural grid point to which other elements are connected
or it may be an unconnected point of the interior of
the rigid element. For convenience in notation, the
reference point is always the first grid point defined
for the element.

Boundary constraints may be specified on any degrees
of freedom of the element. These may take the form
of either complete restraint or imposed displace-
ments. Specification of redundant constraints is
permitted. Checks are incorporated on the admissi-
bility of any set of imposed constraints to ensure no
violation of the rigid body kinematics occurs. If such
a violation is detected, execution of the program is
terminated with an appropriate message to the analyst.

In the presence of boundary restraints or imposed nodal
displacements, the resultant motion of the body is
then expressed in terms of a new set of natural degrees
of freedom which are automatically generated by the
element. The final motion of the body is specified in
the relevant output systems at the individual grid
points.

Under certain circumstances, a rigid body may be
specified which possesses one or more unrestrained
degrees of kinematic freedom. Inclusion of these mo-
tions would result in a singular stiffness matrix.

A check is included to detect this condition and eliminate the unnecessary degrees of freedom from the rigid element reference set. For the simplest case of a fully unrestrained rigid body, the relationship between all components of motions at all gridpoints and in reference set of six independent degrees of freedom at any reference point in the system can be written directly from simple kinematics.

The introduction of considerations of boundary constraints (either as complete restraints or as imposed displacements) linking of the element to adjacent structure by less than six degrees of freedom per gridpoint and the necessity for checking a possibly surplus unrestrained motions of the rigid body considerably complicates the element relationships. All these aspects are handled using rank elimination techniques.

The matrix of complete kinematic relationships between all specified motion (or potential motion) at gridpoints of the rigid body and a basic set of orthogonal components of motion at some reference point in the system is established.

By imposing rank determination techniques selectively on portions of this matrix:

a) the presence of surplus degrees of freedom can be detected through a complete uncoupling of one or more reference degrees of freedom from all specified components of motion of the body. The surplus degrees of freedom are then eliminated using single point constraints.

b) all degrees of freedom associated with the body motion are automatically partitioned into dependent and independent sets. This partitioning reflects the prescribed boundary conditions and ensures that all restrained

motions are included in the independent set. These restraints can then be deleted using single point constraints leaving a reduced set of independent degrees of freedom which describes the residual motion of the restrained rigid element. A check is also incorporated to ensure that any imposed displacement system is admissible (i. e. does not require violation of the rigid body kinematics of the system)

All necessary transformation relationships are retained in order to present to the analyst the full kinematic motion of the rigid body at the output stage.

(iii) Triangular and Trapezoidal Ring Elements

The formulation of the triangular and trapezoidal cross-section ring elements is derived from, and is mathematically consistent with, the formulation described in References 1 and 2. The ring elements provide a powerful tool for the analysis of thick-walled and solid axisymmetric structures of finite length. They may be used to idealize any axisymmetric structure taking into account:

1.   Arbitrary axial variations in geometry.
2.   Axial variation in orientation of material axes of orthotropy.
3.   Radial and axial variations in material properties.
4.   Asymmetric as well as axisymmetric loading systems including pressure and temperature.

The discrete element technique was first applied to the analysis of axisymmetric solids by Clough and Rashid [3]. The formulation of the triangular cross-section ring was extended by Wilson [4] to include

nonaxisymmetric as well as axisymmetric loads.

Wilson's formulation for the asymmetric case was extended in Reference 5 to include orthotropic material properties with variable orientation axes. This extended development was utilized in the derivation of the solid of revolution elements which were provided the NASTRAN program during this study.

The triangular cross-section ring element possesses three nodes and three degrees of freedom at each of these nodes (radial, axial and circumferential displacements) which accounts for a total of 9 degrees of freedom for the element proper. The trapezoidal cross-section ring element is defined by four nodes possessing the same individual degrees of freedom discussed above (totaling 12 element degrees of freedom).

Lumped and consistent mass matrices were developed specifically for the triangular and trapezoidal cross-section ring elements as were thermal load vectors for both of the elements. The triangular and trapezoidal cross-section ring elements were designed to accommodate grid point, pressure and gravity loadings previously existing in the NASTRAN program.

The basic restriction inherent in both elements is that they remain structurally and materially axisymmetric. Circumferential variation of element dimensions (thicknesses) or material properties is not permitted. Both elements are formulated in a cylindrical coordinate system defined by radial (r) , axial (Z) and circumferential (∅) coordinates. Utilizing this coordinate system the radial, tangential and axial displacements of a point (r,Z, ∅) located within the ring element can be expressed as shown below.

$$u(r, z, \phi) = U_o(r, z) + \sum_{n=1}^{m} U_n(r, z) \cos n\phi + \sum_{n=1}^{m} U_n^*(r, z) \sin n\phi$$

$$V(r, z, \phi) = V_o^*(r, z) - \sum_{n=1}^{m} V_n^*(r, z) \cos n\phi + \sum_{n=1}^{m} V_n(r, z) \sin n\phi \quad (1)$$

$$W(r, z, \phi) = W_o(r, z) + \sum_{n=1}^{m} W_n(r, z) \cos n\phi + \sum_{n=1}^{m} W_n^*(r, z) \sin n\phi$$

Equation (1) characterizes the decomposition of the displacement fields into orthogonal components. With the assumption of structural axisymmetry, an uncoupling between harmonics of the element potential and kinetic energy results. This condition is evidenced by the particular example of the strain energy given in Equation 2 below:

$$U_e = U_{eo} + U_{eo}^* + \sum_{n=1}^{m} U_{en} + \sum_{n=1}^{m} U_{en}^* \quad (2)$$

The analysis of a three dimensional axisymmetric structure using the triangular and trapezoidal ring elements is thereby reduced to the solution of a series of two dimensional problems. The resulting analysis procedure is similar to one required by the conical shell of revolution element previously existing in NASTRAN. The existing analytical procedure was therefore implemented in the insertion of the triangular and trapezoidal ring elements into the NASTRAN program.

(iv) Triangular and Quadrilateral Plate Elements
Triangular and quadrilateral plate elements, based upon these presented in Reference 1, were derived and implemented into NASTRAN. These quadrilateral and triangular elements are appropriate for use in the analysis

9

of all types of shell structures, both planar and curved. Although the geometries of the two elements differ, the basic procedures for the derivation of all element relationships are essentially identical. Each element is divided into triangular zones for which displacement functions are assumed for membrane and flexural behavior independently. Membrane displacements are represented by quadratic polynomials while cubic polynomials are used for the transverse displacements. Through the enforcement of interzonal continuity conditions, these assumed displacement functions are then related to each other.

The use of this procedure results in consequent improvement in the element accuracy over similar elements based upon lower order polynomials. The use of independent zonal functions also ensures that the element relationships are independent of the orientation or numbering system of the element.

Under normal circumstances, degrees of freedom are defined at the corner points of the elements and also at the midpoints of the sides. The introduction of midside degrees of freedom, which are consistent with the higher order polynomials used, permits satisfaction of full interelement compatibility along the element boundaries.

At each corner point, five degrees of freedom are defined, corresponding to the three translational components plus rotations about the local x- and y- axes. Rotation about the Z-axis (normal to the plane of the plate) is suppressed. At the midside gridpoints only three degrees of freedom are defined. These are the membrane displacements (u, v) and a rotation (θ_n) about the edge of the element. Since this latter rotational degree of freedom is referenced only to the

edge of the element, some local convention must be
introduced to ensure continuity between adjacent ele-
ments. To achieve this, a local positive direction
is defined for each edge of an element from the lower
to the higher gridpoint number. The direction of
is then defined by a positive vector along each edge.

An option is also provided to use the elements with the
midside degrees of freedom suppressed on any or all sides.
This suppression is accomplished by the analyst insert-
ing a zero in place of a gridpoint identification num-
ber on the element connection card. The result of this
action is to enforce a linear variation of membrane
displacements along the relevant edge.

In using the element, six degrees of freedom are norm-
ally allocated for each gridpoint. The analyst must
then impose boundary constraints to eliminate any un-
defined components. For example, when used in the
flexural analysis of a planar structure, the normal
idsplacement w along with the second and third rota-
tion components must be eliminated at the midside points
through the use of single point constraints. The im-
position of such a w-restraint will not result in a
local dimple in the structure, since the cubic normal
displacement behavior is defined only in terms of the
transverse displacements and rotations of the corners.
In the analysis of curved shells or non-planar struct-
ures, the suppression option should be used on the de-
grees of freedom associated with the mid-points of all
edges between non-coplaner adjacent elements.

While both membrane and flexural action are represented, the
two behaviors are uncoupled within the element and can

be generated separately. Distinct membrane and flex-
ural effective thicknesses are assumed constant over
the plane of the element.

In the thin plate and shell structures typically
modeled by these elements, transverse shear defor-
mations are negligibly small compared with those aris-
ing from membrane and flexural strains. Shear defor-
mations are normally only of significance in thick
plates and shells where the use of the present ele-
ments may be inappropriate. In addition, it has been
shown (Ref. 6) that the use of a direct displacement
approach in the derivation of element stiffness will
lead to an erroneous representation of shear deform-
ability. In view of these considerations and the as-
sociated numerical complexity, transverse shear defor-
mations are not included in these plate elements.

The formulation of the plate elements accommodates
full orthotropy of mechanical and physical material
properties. Orientation of the material axes is
user-specified. A linear generalized Hooke's Law is
used for equations of state. Both plane stress and
plane strain options are available. For thermal load-
ing, the temperature is assumed constant over the sur-
face for membrane behavior and linear through the
thickness for flexure. Temperature referenced ma-
terial properties are assumed constant over the element.
Provision is also made for the use of work equivalent
normal pressure loading which is taken to be constant
over the surface of the elements.

The linear variation in strain within the element, per-
mitted by the assumed displacement functions leads to
similar stress variation. Advantage is taken of this

by generating the stress resultants at the corners
as well as at the center of the element. Inplane
and normal; direct, shear, and bending stress re-
sultants are included. The stress resultants can
be generated with respect to any direction specified
by the user.

The same basic procedure is used for the derivation
of both elements. First, the basic assumed displace-
ment functions are defined followed by the appropri-
ate transformations between the various coordinate
systems are defined. Using a potential energy ap-
proach the membrane and flexural contributions to
the element stiffness matrix are derived, along with
the differential stiffness matrix. The consistent
pressure load matrix is then calculated, followed
by both consistent and lumped mass matrices. Finally,
the stress recovery matrices are presented.

While the derivations of the two elements are essen-
tially similar, some differences do occur due to the
differing basic geometrices. The shape of the quad-
rilateral element is defined by the coordinates of
the four corner points. The element is assumed to be
planar, but since the four corner points may not be
coplanar, the derivation of the characteristics of the
element is based upon an equivalent planar quadrilateral.
This mean plane is defined by the line joining the mid-
points of opposing sides. The four corner points are
then all equidistant from this plane. The equivalent
quadrilateral is obtained by projecting the corner
points directly onto the mean plane. The degree of non-
planarity is indicated to the analyst by a print-out
of the ratio of the distance of the individual corner
points from the mean plane to the average of the lengths
of the quadrilateral diagonals.

If the quadrilateral is non-planar, application of the gridpoint forces derived for the membrane behavior of the planar quadrilateral to the actual gridpoints will result in a violation of moment equilibrium in the system. To eliminate this problem, normal forces are introduced at the corner points. These forces, which have no influence on the membrane behavior of the quadrilateral, are automatically selected to restore the moment equilibrium. In the derivation of the quadrilateral relationships, use is made of a common oblique coordinate system for the definition of the assumed displacement functions in the four constituent triangular zones. The origin of this coordinate system is located at the intersection of the quadrilateral diagonals. It greatly simplifies many of the subsequent computations and is used throughout the element formulation.

The shape of the triangular element is similarly defined by the coordinates of its three corner points. Since three points uniquely define a plane, considerations of warping encountered in the companion quadrilateral element are absent.

The origin of the coordinate system used in the derivation of the element is located at the centroid of the triangle. Since each zonal subtriangle is associated with a different apex angle, no common oblique coordinate system can be defined for all zones, as was possible in the quadrilateral. Hence, the complete derivation is based upon the use of an orthogonal geometric coordinate system.

B.   FINITE ELEMENT HEAT TRANSFER

The analogy between the mechanics and thermodynamics
of solid bodies has been exploited for the develop-
ment of a heat transfer analysis capability in NASTRAN.
The heat transfer analysis in a solid continuum is dis-
cretized by the use of finite elements, as in the case
of structural analysis, reducing the problem to the
solution of a finite set of equilibrium equations in
which the unknowns are temperatures at a discrete grid-
points in the continuum. The general equation govern-
ing discretized heat transfer can be written:

$$[K]\{u\} + [B]\{\dot{u}\} = \{P\} + \{N\}$$

where $\{u\}$ is a vector of gridpoint temperature

$\{P\}$ is a vector of time-dependent applied
heat loads

$\{N\}$ is a vector of temperature-dependent non-
linear heat loads

$[K]$ is the conductivity matrix

$[B]$ is the thermal capacitance matrix

The use of these symbols assists in defining the analogy
between heat transfer and structural analysis. The new
heat transfer analysis capability uses all the normal
analytical tools in NASTRAN provided for structural an-
alysis. The difference being that the matrices $[K], [B], \{P\}$
and $\{N\}$ are generated from thermodynamic rather than
structural properties. Gridpoints are used to locate
temperates, corresponding to displacements in structural
analyses. One major difference between thermal and

and structural analysis is that temperature is a
scalar quantity, whereas displacement is a
vector which may have six components. Thus, in heat
transfer analysis, only one degree-of-freedom is pro-
vided per gridpoint.

The conductivity matrix, $[k]$, and the heat capa-
city matrix, $[B]$, are formed from "element properties,
just as in structural analysis. Volume heat conduc-
tion "elements" are analogous in many ways to struct-
ural elements and they even use the same connection
and property cards. In addition, a part of the heat
conduction matrix may be associated with surface heat
convection or radiation.

The components of the applied heat load vector, $\{P\}$
are associated either with surface heat transfer or
with heat generated inside the volume heat conduction
elements. The vector of nonlinear heat load $\{N\}$
results from surface radiation, from temperature-de-
pendent surface convection, and from temperature-
dependent heat conductivity.

In the case of linear static analysis, $[B]$ and $\{N\}$ are
null, and the governing equation is solved in the same
manner as in linear static structural analysis. The user
has the option to employ both sigle and multipoint con-
straints and many other specialized features normally
associated with structural analysis. New solution tech-
niques are used in nonlinear static analysis and in
transient analysis.

The output of a NASTRAN heat transfer analysis includes
the temperature at gridpoints, the temperature gradients
and heat fluxes within volume heat conduction elements,
and the heat flow into surface elements. The heat flow

into surface elements is further separated into components due to user-prescribed flux, radiation, and convective heat flux.

The heat transfer analysis is performed using the same volume heat conduction elements as are used for structural analysis. These elements are:

Heat Conduction Elements

Type	Elements
Linear	BAR, RØD, CØNRØD, TUBE
Planar	TRMEM, TRIA1A2, QDMEM, QUAD1, QUAD2
Solid of Revolution	TRIARG, TRAPRG
Solid	TETRA, WEDGE, HEXA1, HEXA2

Scalar elements, single point constraints, and multipoint constraints are also available for heat transfer analysis. The same connection and property cards are used for heat transfer and structural analysis. Linear elements have a constant cross-sectional area. For the planar elements, the heat conduction thickness is the membrane thickness. Elements with bending properties, such as BAR and TRIA1, have been included so that the user may use the same elements for the thermal and structural analyses of a given structure. The bending characteristics of the elements do not enter into heat conduction problems. The trapezoidal solid of revolution element, TRPRG, has been generalized to accept general quadrilateral rings (i.e. the top and bottom need not be perpendicular to the Z-axis) for heat conduction only.

The heat conduction elements are composed of constant gradient lines, triangles and tetrahedra. The quadrilaterals are composed of overlapping triangles, and the wedges and hexahedra are formed from sub-tetrahedra in exactly the same way as for the structural case.

Four types of surface heat transfer are provided for both steady state and transient analysis. The types are a prescribed heat flux, a convective heat flux due to the difference between the surface temperature and the local ambient temperature, radiation heat exchange, and a prescribed directed vector heat flux from a distant radiating source. In all cases the heat flux is applied to a surface element defined by gridpoints. There are six distinct types of surface elements:

1. POINT, a flat disc defined by a single gridpoint.
2. LINE, a rectangle defined by two gridpoints.
3. REV, a conical frustrum defined by two grid circles.
4. AREA3, a triangle.
5. AREA4, a quadrilateral.
6. ELCYL, an elliptic cylinder defined by two gridpoints. Its use is restricted to prescribed vector heat flux.

C.   NEW ELEMENTS NOT IMPLEMENTED INTO NASTRAN
(i)   Nonprismatic Beam Element
    a. General Remarks

The nonprismatic beam element extends the capabilities of the present NASTRAN rod and beam elements. An arbitrary polygonal shaped cross section is considered, thus providing for nonuniform beam properties and, secondly, an arbitrary taper is allowed. Multiply connected cross sections are considered by properly sequencing the nodes

which define a cross section. An option was developed
to input all or some of the section properties of any
linearly tapered beam. Asymmetrical cross sections can
also be used thereby permitting coupled bending and shear
deformations in the lateral planes. The only geometric
restriction is that corresponding points on the end sec-
tions are to be connected by straight line segments.
Projections of these line segments onto the element
xy- and xz- planes have common vanishing points on the
x-axis in each plane. This gives a quadratic varia-
tion of cross sectional area and a quartic variation
of area moments of inertia and product moments of in-
ertia. These are written in terms of reference station
properties.

The arbitrary cross section is defined by N nodes form-
ing an N sided polygon referenced to an arbitrary $\bar{x}$,
$\bar{y}$, $\bar{z}$ orthogonal axes system. The nodal coordinates
are used to determine the centroid of the reference
section which in turn defines the reference, x, y, z
axes of the beam element. The reference x-axis is the
loci of all cross section centroids. All element de-
formations are measured with respect to this axis system.
Circular cross sections are treated in like manner ex-
cept that nodal cross sectional coordinates are not
needed.

The stiffness matrix is a 12 x 12 matrix of coeffi-
cients expressing the transformation between the forces
and moments acting at the end points of the beam and the
degrees of freedom at these points. It is first derived
in a local (element) coordinate system, then transformed
to a basic system and includes taper effects, shear de-
formations, shear deformations, shear relief due to taper,

and the effects of an offset shear center. Bending
analyses are based on simple flexure theory wherein
plane sections through the beam, taken normal to the
beam axis, remain plane after the beam is subjected
to loading. No torsional stiffness matrix is derived
for the general cross-section element due to the vast
complexity attendant with such general shapes, how-
ever an option exists to input a torsional stiffness
constant. A torsional stiffness matrix is derived for
the circular cross-section element but warping effects
have been neglected.

Mass matrices for both element cross-sections were de-
rived wherein a mass-lumping procedure was used. Non-
structural mass and an offset center of gravity was
accounted for but cross-sectional rotary effects were
neglected. The mass matrices were also transformed to
a global system.

The existing NASTRAN capabilities to calculate stress
data and associated margins of safety are included in
the development of the subject element. In addition
the presently coded NASTRAN differential stiffness is
used. Additional element matrices include thermal load,
impressed displacement load and line load vectors.

It is imperative to note that the non-prismatic beam
element has all the characteristics of the presently
coded NASTRAN bar and rod elements and will reduce to
those elements for unit value of the taper ratios and
zero value of the shear center offset and mass center
offset.

b. Theoretical Basis

Severn (6) has shown that derivation of a stiffness

matrix including shear effects is erroneous when the displacement method is used and that Pian's (7) method based upon minimum complementary energy) yields the correct results. His method is used to formulate the stiffness matrix including shear deformation and shear relief due to taper.

Analysis begins by expressing the stress distribution $\{\sigma\}$ in terms of a set of coefficients $\{\beta\}$ as follows:

$$\{\sigma\} = [P]\{\beta\} \tag{1}$$

where in this application $\{\sigma\}^T = \lfloor \sigma, \tau \rfloor$. The terms in $[P]$ are functions of the coordinates and shear deformation parameters.

Introduction of the strain-stress relations

$$\{\epsilon\} = [N]\{\sigma\} \tag{2}$$

permits expression of the strain energy as

$$U = \tfrac{1}{2} \int_V \lfloor \sigma \rfloor [N]\{\sigma\}\, dV \tag{3}$$

or

$$U = \tfrac{1}{2} \lfloor \beta \rfloor [H]\{\beta\} \tag{4}$$

where

$$[H] = \int_V [P]^T [N][P]\, dV \tag{5}$$

The prescribed displacements $\{u\}$ at the boundary, A_2, which in the present application is a point, are given in terms of the generalized displacements $\{q\}$ at the nodes in the form

$$\{u\} = \lceil L \rfloor \{q\} \tag{6}$$

where in this instance $\lfloor L \rfloor$ is an identity matrix.
The surface forces $\{S\}$ can be expressed in terms
of the stresses $\{\sigma\}$ by use of

$$\{S\} = [R]\{\beta\} \tag{7}$$

where terms in $[R]$ contain beam characteristics
such as taper ratio and inertia properties.
Use of the above permits the complementary energy
to be written as

$$\Pi_c = \frac{1}{2}\lfloor\beta\rfloor[H]\{\beta\} - \lfloor\beta\rfloor[R]^T\{q\} \tag{8}$$

The condition of minimum complementary energy is
invoked which yields

$$[H]\{\beta\} = [R]^T\{q\} \tag{9}$$

Thus

$$\{\beta\} = [H]^{-1}[R]^T\{q\}. \tag{10}$$

Substituting Equation (10) in Equation (4), one
obtains

$$U = \frac{1}{2}\lfloor q \rfloor[R][H]^{-1}[H][H^{-1}][R]^T\{q\} \tag{11}$$

or

$$U = \frac{1}{2}\lfloor q \rfloor[k]\{q\} \tag{12}$$

where the desired element stiffness matrix is

$$[k] = [R][H]^{-1}[R]^T \tag{13}$$

(ii)　　Exact Shell of Revolution Element and Ring
　　　　　Stiffener

A theoretical formulation for an exact shell of revolution element was developed along with an associated ring stiffener. As was the case of the solid elements of revolution, circumferential axisymmetry with respect to material properties and geometry was assumed in order that the mathematical problem be simplified. An analysis procedure similar to that utilized in the conical shell, triangular ring and trapezoidal ring elements was proposed.

The formulation of the exact shell of revolution element was based on previous formulations by Adelman et al given in References 8 and 9. The theoretical foundation of this shell element is provided by the thin shell theories developed by Novozhilov, Reference 10, and Ambartsumyan, Reference 11. Features of a typical axisymmetric thin shell of revolution which the exact shell of revolution element would be able to represent are:

1) Meridional as well as circumferential curvature of the shell,

2) layering of the shell,

3) accommodation of orthotropic material behavior within each of the layers of the shell where principal directions of orthotropy of the layers are assumed to lie in the meridional, circumferential and normal directions of the shell element,

4) meridional variation of the thickness of each layer with consequent tapering of the shell element as a whole,

5) accommodation of _asymmetric_ mechanical and thermal loadings,

6) provision for the application of circumferential ring stiffeners to the shell structure,

7) provision for accounting for the existence of branch points, branch shells and slope discontinuities.

(iii) Multilayered Triangular and Quadrilateral Flat and Plate Elements

Stiffness and thermal force matrices were derived for multilayered flat triangle and quadrilateral plate finite elements based on the displacement method. Engineering plate theory founded on the Kirchhoff assumption was used. The primary objective for the development of these finite elements is to provide a general analysis capability for laminated structures in which membrane-flexure coupling is not negligible.

Many plate finite element formulations are available and these could be readily extended for application to analysis of laminated structures. Therefore, the criteria for selection of previously developed element formulations were engineering acceptability of analysis results and computational simplicity. For simplicity, the elements developed were limited to five local degrees of freedom at each node point, three displacements and two rotations.

In the derivation of the stiffness matrices for the triangular plate, a linear displacement function was assumed for membrane behavior, while interconsistent cubics were used in two triangular subzones to predict flexural behavior. For the quadrilateral the corresponding assumed displacement functions are quadric and cubic.

24

Based on these assumptions expressions for the stiff-
ness and thermal force matrices for the triangular
and quadrilateral elements were derived. These ex-
pressions included not only the membrane and flexur-
al contributions to the stiffness matrices but also
the membrane-flexural coupling terms. A procedure for
the inclusion of offset gridpoints was derived.

The differential stiffness for flat plate elements is
only a function of plate planform geometry and membrane
stress resultants. Therefore, the displacement func-
tions used to derive the differential stiffness matrices
need not be consistent with those used to develop the
elastic stiffness matrices(12). Consequently, the for-
mulations for the differential stiffness matrices for
the higher order triangular and quadrilateral plate
elements already contained in NASTRAN are valid and
can be used with the two multilayered plate elements in
conjunction with membrane stress resultants computed
therefor. The higher order plate elements in NASTRAN
contain midside degrees of freedom which must be elim-
inated for use in conjunction with the present element
by imposition of the suppress option detailed in the
derivation of these higher order elements.

It has been shown (13) that mass matrices for plate
elements which have been derived using displacement
functions which are not exactly consistent with the
displacement functions used in the stiffness matrix
derivations will still give good analytical results.
Therefore, the consistent mass matrices derived for
the higher order triangle and quadrilateral flat
plate elements in NASTRAN can be used for the present
elements without jeopardizing accuracy.

III. <u>REFERENCES</u>

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